

STANDARD CANONICAL SUPPORT LOCI

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Dedicated to Philippe Ellia on the occasion of his 60th birthday

ABSTRACT. We consider the union of certain irreducible components of cohomological support loci of the canonical bundle, which we call standard. We prove a structure theorem about them and single out some particular cases, recovering and improving results of Beauville and Chen-Jiang. Finally, as an example of application, we extend to compact Kähler manifolds the classification of smooth complex projective varieties with $p_1(X) = 1$, $p_3(X) = 2$ and $q(X) = \dim X$.

1. INTRODUCTION

Let X be a compact Kähler manifold. This paper is concerned with the cohomological support loci of the canonical bundle of X , namely

$$(1.1) \quad V^i(K_X) = \{\mu \in \mathrm{Pic}^0 X \mid h^i(K_X \otimes P_\mu) > 0\}$$

(here P_μ denotes the line bundle on X corresponding to $\mu \in \mathrm{Pic}^0 X$ via the choice of a Poincaré line bundle). One knows the following:

(a) *Every irreducible component W of $V^i(K_X)$ is a translate of a (compact) subtorus $T \subset \mathrm{Pic}^0 X$.* This can be rephrased as follows: we denote $\pi : \mathrm{Alb} X \rightarrow B := \mathrm{Pic}^0 T$ the dual quotient. This defines the composed map $f : X \rightarrow B$ sitting in the commutative diagram

$$(1.2) \quad \begin{array}{ccc} X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb} X \\ & \searrow f & \downarrow \pi \\ & & B \end{array}$$

Then $T = f^* \mathrm{Pic}^0 B$ and, in this notation, item (a) is that, for some $\eta \in \mathrm{Pic}^0 X$,

$$(1.3) \quad W = f^* \mathrm{Pic}^0 B + \eta.$$

(b) *The map f verifies the inequality*

$$(1.4) \quad \dim X - \dim f(X) \geq i.$$

Both (a) and (b) are due to Green and Lazarsfeld ([GL2]).

(c) *The translating points η are torsion modulo $f^* \mathrm{Pic}^0 B$.*

This is due to Simpson [Si] in the projective case (see also [S]) and to Wang in the compact Kähler setting ([W], see also [PPoS] §12), proving a conjecture of Beauville and Catanese.

The understanding of the loci $V^i(K_X)$ is often crucial in the study of irregular compact Kähler varieties but, despite the above powerful theorems, our knowledge of them is still unsatisfactory in some respects. For example, a clear geometric reason for the presence or absence of non-trivial 0-dimensional components is lacking. Another issue is that there is no clear description of the points of finite order appearing as translating points. In fact in [S] 1.1.4-5 Schnell shows an example where, for some i , $V^i(K_X)$ is not *complete*, namely it happens that $W = f^*\text{Pic}^0 B + \eta$ is a component of $V^i(K_X)$ but there is a integer k with $\gcd(k, \text{ord}([\eta])) = 1$ such that $f^*\text{Pic}^0 B + k\eta$ is not (in fact, it is not even contained in $V^i(K_X)$).

In this paper we show that there is a part of the cohomological support loci, which we call *standard*, where the latter problem does not arise and moreover the translating points have a sufficiently clear geometric description. We will use the following terminology: a pair (W, i) , with W irreducible component of $V^i(K_X)$, is called *standard* if equality holds in (1.4), i.e.

$$\dim X - \dim f(X) = i.$$

For example, the only standard pair such that W is 0-dimensional is $(\{\hat{0}\}, \dim X)$.

The union all subvarieties W such that (W, i) is a standard pair will be referred to as *the standard part of $V^i(K_X)$* . As another immediate example, (1.4) implies that the positive-dimensional part of $V^{\dim X-1}(K_X)$ coincides with its standard part (of course they might be empty).

According to some of the current literature, a morphism with connected fibers $g : X \rightarrow Y$ onto a normal compact analytic space Y of maximal Albanese dimension (that is, the image of the Albanese map of Y is equal to $\dim Y$) is called an *irregular fibration*. We denote $\text{Pic}^0(g)$ the kernel of the restriction map of $\text{Pic}^0 X$ to a generic fiber of g . We recall that $\text{Pic}^0(g)$ sits in an extension as

$$(1.5) \quad 0 \rightarrow g^*\text{Pic}^0 Y \rightarrow \text{Pic}^0(g) \rightarrow \Gamma_g \rightarrow 0$$

with Γ_g a finite subgroup of $\text{Pic}^0 X / g^*\text{Pic}^0 Y$ ¹. Thus $\text{Pic}^0(g)$ is the finite union of translated subtori $g^*\text{Pic}^0 Y + \eta$ for $\eta \in \text{Pic}^0 X$ such that its class modulo $g^*\text{Pic}^0 Y$ is in Γ_g .

Theorem A. *For each i the standard part of $V^i(K_X)$ is the union, for all irregular fibrations $g : X \rightarrow Y$ with $\dim X - \dim Y = i$, of*
- *the subvarieties $\text{Pic}^0(g) \setminus (g^*\text{Pic}^0(Y) + N_g)$, where N_g is a finite subgroup of $\text{Pic}^0(g)$, if $\chi(K_{\tilde{Y}}) = 0$, where \tilde{Y} is any desingularization of Y ². If, in addition, the Albanese morphism of Y is surjective then $N_g = \{0\}$.*
- *the subgroups $\text{Pic}^0(g)$ otherwise.*

It is perhaps worth to mention that Theorem A can be also restated as the following Corollary. Given an irregular fibration $g : X \rightarrow Y$, let $i = \dim X - \dim Y$. By Kollár's vanishing theorem (see Theorem 4.1 below) $\text{Pic}^0(g)$ is precisely the locus of $\eta \in \text{Pic}^0 X$ such that $R^i g_*(K_X \otimes P_\eta)$ is non-zero (in fact a torsion-free sheaf of rank one). By deformation-invariance of holomorphic Euler

¹proof: let X_g be a general fiber of g . Then g induces a map $a : X_g \rightarrow K := \ker(\text{Alb } X \rightarrow \text{Alb } Y)$ whose image spans K . Hence the homomorphism $\text{Pic}^0 K \rightarrow \text{Pic}^0 F$ has finite kernel. Then the exact sequence (1.5) follows from the dualization of the exact sequence $0 \rightarrow K \rightarrow \text{Alb } X \rightarrow \text{Alb } Y \rightarrow 0$

²the holomorphic Euler characteristic $\chi(K_{\tilde{Y}})$ does not depend on the particular resolution \tilde{Y} considered. Since one can choose a \tilde{Y} which is Kahler (see e.g. [Ca2], 1.9), $\chi(K_{\tilde{Y}}) \geq 0$ by generic vanishing (see below)

characteristic $\chi(R^i g_*(K_X \otimes P_\eta))$ depends only on $[\eta] \in \text{Pic}^0(g)/g^*\text{Pic}^0 Y$. By generic vanishing (see Theorem 3.1 below) it is always non-negative.

Corollary B. *In the notation above, the set of $[\eta] \in \text{Pic}^0(g)/g^*\text{Pic}^0 Y := \Gamma_g$ such that $\chi(R^i g_*(K_X \otimes P_\eta)) = 0$ is either empty or a subgroup of Γ_g . The latter case holds if and only if $\chi(K_{\tilde{Y}}) = 0$, where \tilde{Y} is any desingularization of Y . If this is the case and, in addition, the Albanese map of Y is surjective then such subgroup is zero.*

In the following particular case, we obtain a more precise description, recovering and extending a well known result of Beauville ([B], Cor.2.3).

Theorem C. *Let p be the maximal index such that $V^p(K_X)$ is positive-dimensional³*

(a) *If $g : X \rightarrow Y$ is an irregular fibration with $\dim X - \dim Y > p$ then Y is bimeromorphic to a complex torus and $\text{Pic}^0(g) = g^*\text{Pic}^0 Y$.*

(b) *The positive-dimensional part of $V^p(K_X)$ is the union, for all irregular fibrations $g : X \rightarrow Y$ with $\dim X - \dim Y = p$, of:*

- *the subvarieties $\text{Pic}^0(g) \setminus g^*\text{Pic}^0(Y)$ if Y is bimeromorphic to a complex torus,*
- *the subgroups $\text{Pic}^0(g)$ otherwise.*

Beauville's aforementioned result is statement (b) for $V^{\dim X - 1}(K_X)$ (note that $p = \dim X - 1$ if $V^{\dim X - 1}(K_X)$ is positive-dimensional). In this case the positive-dimensional components are induced by fibrations onto smooth curves of genus ≥ 1 . Even in this case our proof is different from Beauville's. There some questions naturally connected to the above results:

(a) It would be interesting to give a geometric description for the groups $\text{Pic}^0(g)$ associated to irregular fibrations (and of the subgroups N_g appearing in the statement of Theorem A). When the base Y is a curve, $\text{Pic}^0(g)$ is completely described in terms of the multiple fibers of g (Beauville [B]).

(b) Topological invariance of irregular fibrations of compact Kahler manifolds and their number (up to equivalence), as well as their relation with the fundamental group. This matter is well understood when the basis of the fibration is a curve of genus ≥ 2 by work of Siu, Beauville and Arapura (see [Ca1], [B], [A], [Si]). In Beauville and Arapura's treatment the main ingredient is the above mentioned result of Beauville. In view of Theorem C(b) it is natural to ask for similar results for irregular fibrations over normal analytic spaces $X \rightarrow Y$ of arbitrary dimension, at least when $\chi(K_{\tilde{Y}}) > 0$. Interestingly, Catanese ([Ca1]) proved – with a different approach – the topological invariance of the existence of irregular fibrations $X \rightarrow Y$ such that the Albanese map of Y is non-surjective (note that when Y is a smooth curve the condition $\chi(K_Y) > 0$, i.e. $g(Y) \geq 2$, is equivalent to the non-surjectivity of the Albanese map of Y , but this is not anymore true in higher dimension).

(c) A conjecture of M. Popa predicts that all loci $V^i(K_X)$ are derived-invariants. In view of Theorem C this would imply that the integer p is a derived invariant, as well as all (equivalence classes of) irregular fibrations $g : X \rightarrow Y$ such that $\dim X - \dim Y = p$ (except those such that Y is bimeromorphic to a complex torus and $\text{Pic}^0(g) = g^*\text{Pic}^0 Y$). Again, this is (partly) known when the base Y is a smooth curve ([Po], [LPo]).

³if $V^i(K_X)$ is empty or zero-dimensional for all i we define $p = -\infty$

(d) It would be interesting to find a larger part of canonical cohomological support loci admitting a description similar to Theorem A.

Another consequence of Theorem A is

Corollary D. *Assume that X has maximal Albanese dimension (that is: $\dim \text{alb}_X(X) = \dim X$) and that $V^0(K_X)$ is a proper subvariety of $\text{Pic}^0 X$. Then $V^0(K_X)$ is the union, for all irregular fibrations $g : X \rightarrow Y$ such that $\dim X - \dim Y = \dim \text{Alb } X - \dim \text{Alb } Y$, of:*

- *the subvarieties $\text{Pic}^0(g) \setminus (g^* \text{Pic}^0(Y) + N_g)$, where N_g is a finite subgroup of $\text{Pic}^0(g)$, if $\chi(K_{\tilde{Y}}) = 0$ (where \tilde{Y} is any desingularization of Y). If, in addition, the Albanese morphism of Y is surjective then $N_g = \{0\}$.*
- *the subgroups $\text{Pic}^0(g)$ otherwise.*

A weaker statement along these lines was proved in [P], 4.3 . Corollary D is also a strengthening of a result of [CJ], (Th. 3.5) (in turn generalized in [PPoS] Cor. 16.2), asserting that $V^0(K_X)$ is invariant with respect to the natural involution of $\text{Pic}^0 X$. Note that from Theorems A and C, and Corollary D it follows that the loci in question are *complete* in the above sense.

All proofs are based on Hacon's generic vanishing theorem for higher direct images, often combined with Kollár's decomposition for the derived direct image of the canonical sheaf. A key tool for the proof of Theorem A is a sharper version of Hacon's theorem introduced by J. A. Chen and Z. Jiang. We will refer to that as the Chen-Jiang decomposition. Hacon's and Chen-Jiang's theorem was extended to the compact Kähler setting and to higher direct images in [PPoS].

Results as the above are useful in applications concerning the geometry and the classification of irregular compact Kähler manifolds. In what follows we will denote $p_i(K_X) = h^0(K_X^i)$ the plurigeners of a compact Kähler manifold X . For example, already the aforementioned invariance of $V^0(K_X)$ under the natural involution of $\text{Pic}^0 X$ was a key point in the proof that complex tori are classified by their irregularity and first two plurigeners ([PPoS], Th.B) (this was a theorem of Chen-Hacon in the algebraic case ([CH1])). In the last section, which is somewhat independent, we provide a related application. In fact, after complex tori, it is natural to aim at the classification of compact Kähler irregular manifolds with low plurigeners. In the projective case this has been pursued by various authors, see [CH2], [CH3], [H2], [HPa1], [J]. Still under the conditions $q(X) = \dim X$ and $p_1(X) \neq 0$, it turns out that the next lowest condition on plurigeners is $p_3(X) = 2$. We confirm the classification of such varieties in the projective case, due to Hacon-Pardini ([HPa1] Th.4)⁴.

Theorem E. *Let X be a compact Kähler manifold with $q(X) = \dim X$, $p_1(X) \neq 0$ and $p_3(X) = 2$. Then $\text{Alb } X$ has a quotient (with connected fibers) $\pi : X \rightarrow E$ with E elliptic curve, and X is bimeromorphic to the ramified double cover of $a : X \rightarrow \text{Alb } X$ such that*

$$a_* K_X = \mathcal{O}_{\text{Alb } X} \oplus (\pi^* \mathcal{O}_E(p) \otimes P_\eta),$$

where p is a point of E and η is an element of order two of $\text{Pic}^0 X \setminus \pi^* \text{Pic}^0 E$.

⁴strictly speaking the proof of this application uses only the well known theorem of Beauville mentioned above, which is now a particular case of Theorem C. However we included it in this paper because it is suggestive about the possible use of Theorems A and C when dealing with this sort of problems

It should be mentioned that the result of [HPa1] is stronger, since it works without the hypothesis $p_1(X) \neq 0$, which, in our treatment, is used to ensure the surjectivity of the Albanese map. I will come back to this point in the future. However, apart from this issue, the argument here seems to be simpler and more self-contained. Hopefully this method will find more application to the classification of irregular compact Kähler manifolds.

The paper is organized as follows: there are five background sections, containing material probably known to the experts, but not entirely found in the literature. The reader can use them as a glossary, starting directly from §6. Although Theorem C is essentially a more precise version of a particular case Theorem A, as a matter of expository preference we prove it directly in §6, with a simpler and more self-contained argument. Theorem A and the other corollaries are proved in §7. Theorem E is proved in §8.

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2. BACKGROUND MATERIAL: GV, M-REGULAR, EXTREMAL COMPONENTS

Let X be a compact Kähler manifold and $a : X \rightarrow A$ a morphism to a complex torus. Given a coherent sheaf \mathcal{F} on X one can consider its cohomological support loci (with respect to a)

$$(2.1) \quad V_a^i(X, \mathcal{F}) = \{\alpha \in \text{Pic}^0 A \mid h^i(X, \mathcal{F} \otimes a^* P_\alpha) > 0\}$$

As for cohomology groups, we will often suppress X from the notation, writing simply $V_a^i(\mathcal{F})$. The cohomological support loci defined in (1.1) are particular cases of the above, since $\text{Pic}^0 X = \text{alb}^* \text{Pic}^0(\text{Alb } X)$ via the Albanese map $\text{alb} : X \rightarrow \text{Alb } X$. If this is not cause of confusion, when $a = \text{alb}$ we suppress it from the notation, and simply write $V^i(\mathcal{F})$.

A coherent sheaf \mathcal{F} on X is said to be *GV* (with respect to a) if

$$\text{codim}_{\text{Pic}^0 A} V_a^i(\mathcal{F}) \geq i \quad \text{for all } i.$$

In particular, this implies that the loci $V_a^i(\mathcal{F})$ are strictly contained in $\text{Pic}^0 A$ for all $i > 0$. Hence $\chi(\mathcal{F}) \geq 0$, and $\chi(\mathcal{F}) > 0$ if and only if $V_a^0(\mathcal{F}) = \text{Pic}^0 A$.

A GV sheaf \mathcal{F} (with respect to a) is said to be *M-regular* if the inequalities are strict for $i > 0$, namely $\text{codim}_{\text{Pic}^0 A} V_a^i(\mathcal{F}) > i$ for $i > 0$. Therefore, the difference between GV and M-regular is the presence of subvarieties of V^i of codimension i for some $i > 0$. Henceforth we will refer at them as *extremal components*. This difference is best appreciated via the Fourier-Mukai transform associated to a Poincaré line bundle. We refer to the surveys [PPo3] and [P] or to the papers [CJ], [PPoS] for a thorough discussion of this aspect. Here we will give just a minimal account.

In the rest of this section we will assume that \mathcal{F} is coherent sheaf on the complex torus A (and the morphism a is simply the identity). Let \mathcal{P} be a Poincaré line bundle on $A \times \text{Pic}^0 A$. Let

$$\Phi_{\mathcal{P}} : \mathbf{D}(\text{coh}(\mathcal{O}_A)) \rightarrow \mathbf{D}(\text{coh}(\mathcal{O}_{\text{Pic}^0 A}))$$

be the Fourier-Mukai functor associated to \mathcal{P} . As it is well known, this is an equivalence ([M], see also [Hu], and [BeBlPa] for the case of non-algebraic complex tori). We consider also the (unshifted) dualizing functor $\mathbf{D}(\text{coh}(\mathcal{O}_A)) \rightarrow \mathbf{D}(\text{coh}(\mathcal{O}_A))$ defined by $\mathcal{F}^\vee = \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_A)$.

Theorem 2.1. *Let $q = \dim A$. Let \mathcal{F} be a coherent sheaf on a complex torus A .*

(a) *\mathcal{F} is a GV sheaf if and only if $\Phi_{\mathcal{P}}(\mathcal{F}^\vee)$ is a sheaf in cohomological degree q , denoted $\widehat{\mathcal{F}}^\vee[-q]$. The support of $\widehat{\mathcal{F}}^\vee[-q]$ is $-V^0(\mathcal{F})$.*

(b) *A GV sheaf \mathcal{F} is M-regular if and only if the sheaf $\widehat{\mathcal{F}}^\vee$ is torsion-free. If \mathcal{F} is not M-regular, to each extremal component for \mathcal{F} , say W , corresponds a torsion subsheaf of $\widehat{\mathcal{F}}^\vee$ (supported on $-W$) and conversely.*

As mentioned above, this is a well known fact, see e.g. [PPo3] and [P]. A consequence of Theorem 2.1 is the following non-vanishing result, see e.g. [P] Lemma 1.12 where it is stated only in the algebraic case, but its proof works for complex tori as well.

Corollary 2.2. *Let \mathcal{F} be a non-zero GV sheaf on a complex torus A . Then*

(a) *$V^0(\mathcal{F}) \neq \emptyset$.*

(b) *if \mathcal{F} is a M-regular then $V^0(\mathcal{F}) = \text{Pic}^0 A$ and $\chi(\mathcal{F}) > 0$.*

Here is a basic example of GV but non-M-regular sheaf occurring frequently in what follows.

Example 2.3. [Pullback of M-regular sheaves on quotients] Let $\pi : A \rightarrow B$ be a surjective morphism of complex tori, with $\dim A - \dim B = m > 0$, and let \mathcal{F} be a M-regular sheaf on B . Then for all $\alpha \in \text{Pic}^0 A$ the sheaf (on A) $\pi^*(\mathcal{F}) \otimes P_\alpha^{-1}$ is GV but not M-regular. Indeed, supposing for simplicity that π has connected fibers, for $j \leq m$ we have that

$$(2.2) \quad R^j \pi_*(\pi^* \mathcal{F} \otimes P_\gamma) = \begin{cases} (\mathcal{F} \otimes P_\beta)^{\oplus \binom{m}{j}} & \text{for } P_\gamma = \pi^* P_\beta \text{ with } \beta \in \text{Pic}^0 B \\ 0 & \text{otherwise} \end{cases}$$

Since \mathcal{F} is assumed to be M-regular we have that

$$(2.3) \quad V^k(B, \mathcal{F}) = \begin{cases} \text{Pic}^0 B & \text{for } k = 0 \\ \subsetneq \text{Pic}^0 B & \text{otherwise.} \end{cases}$$

Therefore, combining (2.2), (2.3), projection formula and the Leray spectral sequence we get that

$$V^k(A, p_i^*(\mathcal{F}) \otimes P_\alpha^{-1}) = \pi^* \text{Pic}^0 B + \alpha \quad \text{for } k = 0, \dots, m$$

for all $\alpha \in \text{Pic}^0 A$. In particular $V^m(A, \pi^*(\mathcal{F}) \otimes P_\alpha^{-1})$ has codimension m , hence $\pi^* \mathcal{F} \otimes P_\alpha^{-1}$ is GV but it is not M-regular. A similar computation shows that $\pi^* \text{Pic}^0 B + \alpha$ is the only extremal component for the sheaf $\pi^*(\mathcal{F}) \otimes P_\alpha^{-1}$.

This is perhaps more suggestively seen from the Fourier-Mukai point of view. Here we will use the following basic fact about Fourier-Mukai transforms associated to Poincaré line bundle on complex tori. Let $\pi : A \rightarrow B$ be a quotient of complex tori, and let $\hat{\pi} : \text{Pic}^0 B \rightarrow \text{Pic}^0 A$ be the dual homomorphism. Then we have the following natural isomorphism of functors (see [CJ] Prop. 2.3, where it is stated for abelian varieties, but the proof works for complex tori without changes)

$$(2.4) \quad \Phi_{\mathcal{P}_A} \circ \pi^* \cong \hat{\pi}_* \circ \Phi_{\mathcal{P}_B}[\dim B - \dim A]$$

Gong back to the subject of the present Example, we know from Theorem 2.1(b) that the Fourier-Mukai transform on B of \mathcal{F}^\vee is a torsion-free sheaf, say \mathcal{G} , in cohomological degree $\dim B$ on $\text{Pic}^0 B$.

By (2.4) the Fourier-Mukai transform on A of $(\pi^*\mathcal{F})^\vee$ is the torsion sheaf, in cohomological degree $\dim A$, consisting of the torsion-free sheaf \mathcal{G} on $\pi^*\mathrm{Pic}^0 B$ seen as a torsion sheaf on $\mathrm{Pic}^0 A$.

Similarly the Fourier-Mukai transform on A of $\pi^*(\mathcal{F} \otimes P_\alpha^{-1})^\vee$ is the torsion-free sheaf on $\pi^*\mathrm{Pic}^0 B - [\alpha] = -V_0(\pi^*(\mathcal{F}) \otimes P_{\alpha^{-1}})$, seen as a torsion sheaf on $\mathrm{Pic}^0 A$.

3. BACKGROUND MATERIAL: GENERIC VANISHING THEOREM AND CHEN-JIANG DECOMPOSITION, I

The idea of generic vanishing is due to Green and Lazarsfeld ([GL1] and [GL2]). Since then their theorems have been extended in various directions. One of these is generic vanishing for higher direct images of canonical sheaves, due to Hacon, [H1]. The most updated version is Theorem 3.1 below. The idea of a decomposition as in the statement was introduced for $i = 0$ in the projective case in [CJ]. This was extended to all i , also in the compact Kähler setting, was proved in [PPoS] using the theory of Hodge modules. It will be referred to as *Chen-Jiang decomposition*.

Theorem 3.1. ([H1], [CJ], [PPoS]) *Let $f : X \rightarrow A$ be a morphism from a compact Kähler manifold to a complex torus. Let η be a point of finite order of $\mathrm{Pic}^0 X$. Then, for all i ,*

$$R^i f_*(K_X \otimes P_\eta) = \bigoplus_k \pi_k^*(\mathcal{F}_k) \otimes P_{\alpha_k}^{-1}$$

where: each $\pi_k : A \rightarrow B_k$ is a surjective morphism of complex tori with connected fibers, \mathcal{F}_k is a M -regular coherent sheaf supported on a projective subvariety of B_k , and α_k is a point of finite order of $\mathrm{Pic}^0 A$. In particular, $R^i f_*(K_X \otimes P_\eta)$ is a GV sheaf on A .

Remark 3.2. (Chen-Jiang summands.) (a) Note that the homomorphisms π_k can include the identity of A . By Corollary 2.2, this happens when $\chi(R^i f_*(K_X \otimes P_\eta)) > 0$, which is the generic rank of the M -regular summand. Since the support of $R^i f_*(K_X \otimes P_\eta)$ is torsion-free on $f(X)$ (Theorem 4.1 below) by Theorem 3.1 $f(X)$ is a projective variety in this case.

(b) By Example 2.3 other summands in the Chen-Jiang decomposition appear if and only if $R^i f_*(K_X \otimes P_\eta)$ is not M -regular. More precisely: *there is exactly one of them for each pair (m, W) with $m > 0$ and W a extremal component of $V^m(R^i f_*(K_X \otimes P_\eta))$ (here $m = \dim A - \dim B_k$).*

The important results summarized in the following Corollary are originally due to Ein and Lazarsfeld ([EL]). In the present treatment they follow at once from the Chen-Jiang decomposition and Example 2.3

Corollary 3.3. (Ein-Lazarsfeld) (a) *Assume that $V^0(R^i f_*(K_X \otimes P_\eta))$ is a proper subvariety of $\mathrm{Pic}^0 X$ (i.e. $\chi(R^i f_*(K_X \otimes P_\eta)) = 0$). Then, for each $j > 0$, every component of codimension j of $V^0(R^i f_*(K_X \otimes P_\eta))$ is also an extremal component, namely a j -codimensional component of $V^j(R^i f_*(K_X \otimes P_\eta))$. In particular, if there is an isolated point in $V^0(R^i f_*(K_X \otimes P_\eta))$ then $V^{\dim A}(R^i f_*(K_X \otimes P_\eta))$ is not empty, hence the map f is surjective.*

(b) *Extremal components of $R^i f_*(K_X \otimes P_\eta)$ are subtori-translates of the form $\pi_k^*(\mathrm{Pic}^0 B_k) + \alpha_k$ such that the fibers of the map $\pi_k \circ f : X \rightarrow B_k$ surject on the fibers of the homomorphism $\pi_k : A \rightarrow B_k$. Equivalently: $\dim f(X) - \dim \pi_k(f(X)) = \dim A - \dim B_k$.*

The Fourier-Mukai meaning of the Chen-Jiang decomposition is summarized in the following

Remark 3.4. (FM transform and Chen-Jiang decomposition.) Let $i \leq \dim X - \dim f(X)$, and denote $\mathcal{R}_i = R^i f_*(K_X \otimes P_\eta)$. Assuming that it is non-zero, the combination of Theorems 3.1 and 2.1 tells that:

- (i) the FM-transform of \mathcal{R}_i^\vee is a sheaf in cohomological degree $\dim A$: $\widehat{\mathcal{R}}_i^\vee[-\dim A]$;
- (ii) the sheaf $\widehat{\mathcal{R}}_i^\vee$ is the direct sum of its torsion part and its torsion-free part (one of them can be zero);
- (iii) the torsion part of $\widehat{\mathcal{R}}_i^\vee$ (if any) is the direct sum of torsion-free sheaves on translates of subtori $\pi_k^* \text{Pic}^0 B_k - [\alpha_k]$, seen as sheaves on $\text{Pic}^0 A$. These sheaves are the translates by $-\alpha_k$ of the transforms on B_k of \mathcal{F}_k^\vee . They are in 1-1 correspondence with the extremal components.

Remark 3.5. (Uniqueness of the Chen-Jiang decomposition.) From the previous Remark it follows that the Chen-Jiang decomposition is essentially *canonical*: the sheaves $\pi_k^* \mathcal{F}_k \otimes P_{\alpha_k}^{-1}$ are essentially unique. In fact – via the inverse FM functor $\mathbf{D}(\text{coh}(\mathcal{O}_{\text{Pic}^0 A})) \rightarrow \mathbf{D}(\text{coh}(\mathcal{O}_A))$ – their duals are the transforms respectively of the torsion-free part of $\widehat{\mathcal{F}}^\vee$ and of the components of the torsion part of $\widehat{\mathcal{F}}^\vee$. In particular, their supports, namely the translated subtori $\pi_k^* \text{Pic}^0 B_k + \alpha_k$ are uniquely determined (up to reordering)⁵.

4. BACKGROUND MATERIAL: GENERIC VANISHING THEOREM AND CHEN-JIANG DECOMPOSITION, II

4.1. Kollár decomposition. This is the other essential tool. We state it only in the version we will need

Theorem 4.1. *Let $f : X \rightarrow Y$ be a proper morphism from a compact Kähler manifold to a reduced and irreducible analytic space, and let η be a torsion point of $\text{Pic}^0(X)$. Then, in the derived category $\mathbf{D}(\text{coh}(\mathcal{O}_Y))$,*

$$\mathbf{R}f_*(K_X \otimes P_\eta) = \bigoplus_j (R^j f_*(K_X \otimes P_\eta))[-j]$$

Moreover, if f is surjective, then $R^j f_(K_X \otimes P_\eta)$ is torsion-free for every $j \geq 0$. In particular, it vanishes for $j > \dim X - \dim Y$. In general $R^j f_*(K_X \otimes P_\eta)$ vanishes for $j > \dim X - \dim f(X)$.*

This theorem is due to Kollár in the case when Y is projective. When Y is an analytic space the degeneration of the Leray spectral sequence at E_2 and the torsion-freeness are due to Takegoshi [T]. Saito [Sa2] greatly generalized the results of Kollár, using the theory of Hodge modules. Using [Sa1] his treatment works also in the analytic setting, as stated in Theorem 4.1 (see also [PPoS]).

A standard consequence, proved in [K2] Thm. 3.4 (in the algebraic case, however the same proof goes over in the analytic setting) is that the previous statement is still valid replacing the pair (X, K_X) with the pair $(Y, R^j f_*(K_X \otimes P_\eta))$, for any $j \leq \dim X - \dim Y$:

⁵ the surjective homomorphisms with connected fibres $\pi_k : A \rightarrow B_k$ are not uniquely determined. However one can arrange them in such a way that π_k factorizes through π_h if $\pi_k^* \text{Pic}^0 B_k$ is contained in $\pi_h^* \text{Pic}^0 B_h$

Corollary 4.2. *Let X be a compact Kähler manifold and let Y, Z be reduced and irreducible analytic spaces. Let $f : X \rightarrow Y$ and $a : Y \rightarrow Z$ proper surjective morphisms. Let $\eta \in \text{Pic}^0 X$ and $\beta \in \text{Pic}^0 Y$ be torsion line bundles. Then for all i :*

(a)

$$R^i(a \circ f)_*(K_X \otimes P_\eta) = \bigoplus_j R^j a_* R^{i-j} f_*(K_X \otimes P_\eta)$$

(b)

$$\mathbf{R}a_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\beta) \simeq \bigoplus_j (R^j a_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\beta))[-j]$$

in the derived category $\mathbf{D}(\text{coh}(\mathcal{O}_Z))$.

(c) For all i and j the sheaf $R^j a_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\beta)$ is torsion-free. In particular, it vanishes for $j > \dim Y - \dim Z$.

The proof is as Thm. 3.4 of *loc cit* (note that, under the hypotheses of the Theorem, $P_\eta \otimes f^* P_\beta$ is a torsion line bundle.). Note also that item (iv) of *loc cit*, which makes sense only in the projective case, is not used to prove the other assertions. Combining Theorem 3.1 with Corollary 4.2 we obtain

Theorem 4.3. *Let X be a compact Kahler manifold, $f : X \rightarrow A$ a morphism to a complex torus, and $\pi : A \rightarrow B$ a homomorphism of complex tori. Let also $\eta \in \text{Pic}^0 X$ and $\beta \in \text{Pic}^0 B$ be points of finite order. Then, for each i and j*

$$R^j \pi_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\beta) = \bigoplus_k \sigma_k^*(\mathcal{G}_k) \otimes P_{\gamma_k}^{-1}$$

where: $\sigma_k : B \rightarrow C_k$ is a surjective morphism of complex tori with connected fibers, each \mathcal{G}_k is a M -regular coherent sheaf supported on a projective subvariety of the complex torus C_k , and γ_k is a point of finite order of $\text{Pic}^0 B$. In particular $R^j \pi_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\beta)$ is a GV sheaf on B .

Proof. By Theorem 3.1 and Corollary 4.2(a)

$$(4.1) \quad R^{i+j}(\pi \circ f)_*(K_X \otimes P_\eta \otimes f^* P_\beta) = \bigoplus_{h+l=i+j} R^h \pi_*(R^l f_*(K_X \otimes P_\eta) \otimes P_\beta) = \bigoplus_k \sigma_k^*(\mathcal{F}_k) \otimes P_{\alpha_k}^{-1}$$

We have to prove that the summands of the Chen-Jiang decomposition on the right split in such a way to provide Chen-Jiang decompositions of the individual summands in the middle. This follows from the uniqueness and Fourier-Mukai-theoretic meaning of the Chen-Jiang decomposition (Remarks 3.4 and 3.5). \square

5. BACKGROUND MATERIAL: COMPONENTS OF COHOMOLOGICAL SUPPORT LOCI

5.1. **Components of $V^i(K_X)$.** How do components of $V^i(K_X)$ arise? Recalling the notation of the Introduction we have

$$(5.1) \quad \begin{array}{ccc} X & \xrightarrow{alb_X} & Alb X \\ & \searrow f & \downarrow \pi \\ & & B \end{array}$$

As we know from (a) and (c) of the Introduction, a component W of $V^i(K_X)$ is of the form $f^*(Pic^0 B) + \eta$, with η a point of finite order of $Pic^0 X$. This means that a point $\alpha \in Pic^0 X$ belongs to W if and only if $P_\alpha = P_\eta \otimes f^* P_\beta$ for some $\beta \in Pic^0 B$. Hence, in the notation of §1,

$$f^* Pic^0 B = V_f^i(K_X \otimes P_\eta)$$

By Koll ar decomposition (Theorem 4.1) and projection formula

$$V_f^i(K_X \otimes P_\eta) = f^* V^0(R^i f_*(K_X \otimes P_\eta)) \cup f^* V^1(R^{i-1} f_*(K_X \otimes P_\eta)) \cup \dots$$

By Theorem 3.1, all loci in the right hand side are proper subvarieties of $Pic^0 B$ except for the first one. It follows that

$$W = f^* V^0(R^i f_*(K_X \otimes P_\eta))$$

By (b) of Theorem 4.1 $R^i f_*(K_X \otimes P_\eta)$ vanishes for $\dim X - \dim f(X) < i$. This proves the basic inequality (1.4):

$$\dim X - \dim f(X) \geq i.$$

Summarizing, so far we got that:

a component W of $V^i(K_X)$ is always of the form

$$W = f^* V^0(R^i f_*(K_X \otimes P_\eta)) + \eta = f^* Pic^0 B + \eta$$

where $f : X \rightarrow B$ is a morphism to a complex torus such that $\dim X - \dim f(X) \geq i$.

Next, we consider the Stein factorization of the morphism f of (5.1)

$$(5.2) \quad \begin{array}{ccc} X & \xrightarrow{alb_X} & Alb X \\ \downarrow g & \searrow f & \downarrow \pi \\ Y & \xrightarrow{a} & B \end{array}$$

Lemma 5.1. *In the above setting B must be $Alb Y$ and, up to translation, $a = alb_Y$. It follows that: all components of $V^i(K_X)$ are translates of subtori of the form*

$$g^* V^0(R^i g_*(K_X \otimes P_\eta)) + \eta = g^* Pic^0 Y + \eta$$

for pairs (g, η) such that:

- $g : X \rightarrow Y$ is an irregular fibration with $\dim X - \dim Y \geq i$;
- η is a torsion point of $Pic^0 X$ such that $V^0(R^i g_*(K_X \otimes P_\eta)) = Pic^0 Y$, i.e. $\chi(R^i g_*(K_X \otimes P_\eta)) > 0$.

Conversely, given a pair (g, η) as above, $g^ Pic^0 Y + \eta$ is contained in $V^i(K_X)$.*

Proof. Since both maps a and π factor through $\text{Alb } Y$, diagram (5.2) is factorised as follows

$$(5.3) \quad \begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb } X \\ \downarrow g & \searrow f_Y & \downarrow \pi_{\text{Alb } Y} \\ Y & \xrightarrow{\text{alb}_Y} & \text{Alb } Y \\ \searrow a & & \downarrow \sigma \\ & & B \end{array} \quad \begin{array}{c} \nearrow f \\ \searrow \pi \end{array}$$

By Corollary 4.2(c) $R^h \text{alb}_{Y*}(R^i g_*(K_X \otimes P_\eta)) = 0$ for $h > 0$. Therefore $R^i f_{Y*}(K_X \otimes P_\eta) = \text{alb}_{Y*} R^i g_*(K_X \otimes P_\eta)$. Hence, by Theorem 3.1 and an easy Leray spectral sequence $R^i g_*(K_X \otimes P_\eta)$ is a GV sheaf (with respect to alb_Y) and $\chi(R^i g_*(K_X \otimes P_\eta)) = \chi(R^i f_{Y*}(K_X \otimes P_\eta)) \geq 0$. We claim that the strict inequality holds, that is: $V^0(R^i f_{Y*}(K_X \otimes P_\eta)) = \text{Pic}^0 Y$. This implies that $g^*(\text{Pic}^0 Y) + \eta$ is contained in $V^i(K_X)$ and contains the component W , hence they must be equal. Moreover $\text{Pic}^0 B = \text{Pic}^0 Y$. Therefore the claim proves the Lemma.

To prove what claimed we argue as follows. We know that $V^0(R^i f_{Y*}(K_X \otimes P_\eta)) = \text{Pic}^0 B$. If $V^0(R^i f_{Y*}(K_X \otimes P_\eta))$ is strictly contained in $\text{Pic}^0 Y$ then $\sigma^* \text{Pic}^0 B$ must be a component of $V^0(R^i f_{Y*}(K_X \otimes P_\eta))$, say of codimension j . Then we know by Corollary 3.3 that $\sigma^* \text{Pic}^0 B$ is also a component of $V^j(R^i f_{Y*}(K_X \otimes P_\eta)) = V_{\text{alb}_Y}^j(R^i g_*(K_X \otimes P_\eta))$. But, as the map a is finite, $R^i(a \circ g)_*(K_X \otimes P_\eta) = a_* R^i g_*(K_X \otimes P_\eta)$, as above. Therefore, again by an easy Leray spectral sequence, $\text{Pic}^0 B = V^j(R^i(a \circ g)_*(K_X \otimes P_\eta))$, hence $R^i(a \circ g)_*(K_X \otimes P_\eta)$ is not a GV sheaf on the complex torus B , in contradiction with Theorem 3.1.

The last assertion follows by the Kollár decomposition. \square

5.2. Components of $V^i(R^j g_*(K_X \otimes P_\eta))$. The previous Lemma relates the loci $V^i(K_X)$ to the loci $V^0(R^i f_{Y*}(K_X \otimes P_\eta))$ for suitable morphisms to complex tori $f : X \rightarrow B = \text{Alb } Y$ or, what is the same, to the loci $V^0(R^i g_*(K_X \otimes P_\eta))$, where in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb } X \\ \downarrow g & \searrow f & \downarrow \pi \\ Y & \xrightarrow{\text{alb}_Y} & \text{Alb } Y \end{array}$$

g is the Stein factorization of the morphism f . More generally, it is useful to describe in a similar way the components of the cohomological support loci $V^r(R^i g_*(K_X \otimes P_\eta))$, for all i and r . This is the content of part (a) of the following Lemma. Part (b) provides an explicit description of extremal components.

Lemma 5.2. *In the above notation, let η be a point of finite order of $\text{Pic}^0 X$.*

(a) *For all integers r and i the components of $V^r(R^i g_*(K_X \otimes P_\eta))$ are of the form*

$$(5.4) \quad h^* V^0(R^r h_*(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha) + \alpha = h^* \text{Pic}^0 Z + \alpha$$

for pairs (h, α) such that:

- $h : Y \rightarrow Z$ is an irregular fibration with $\dim Y - \dim Z \geq r$;

- α is a point of finite order of $\text{Pic}^0 Y$ such that $V^0(R^r h_*(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha)) = \text{Pic}^0 Z$, i.e. $\chi(R^r h_*(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha)) > 0$.

Conversely, given a pair (h, α) as above, $h^* \text{Pic}^0 Z + \alpha$ is contained in $V^r(R^i g_*(K_X \otimes P_\eta))$.

(b) Extremal components for $R^i g_*(K_X \otimes P_\eta)$, i.e. components of $V^r(R^i g_*(K_X \otimes P_\eta))$ of codimension r for some r , are of the form (5.4) for pairs (h, α) , where $h : Y \rightarrow Z$ is an irregular fibration such that $\dim Y - \dim Z = r = q(Y) - q(Z)$ (here $q(Y)$ and $q(Z)$ denote $\dim \text{Alb } Y$ and $\dim \text{Alb } Z$) and α is such that $\chi(R^r h_*(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha)) > 0$.

Conversely, given a pair (h, α) as above, $h^* \text{Pic}^0 Z + \alpha$ is a component of $V^r(R^i g_*(K_X \otimes P_\eta))$ of codimension r .

Proof. We recall that the subtorus Theorem (namely items (a) and (c) of the Introduction) holds as well for the sheaves $R^i g_*(K_X \otimes P_\eta)$ (or, equivalently for the sheaves $R^i f_*(K_X \otimes P_\eta)$). For extremal components this follows at once from Theorem 3.1 and Example 2.3, but in fact it holds more generally for all components, see e.g. [HPa2] Thm 2.2(b)⁶. Thus all ingredients for the proof of Lemma 5.1 (vanishing theorem, Koll r decomposition, subtorus theorem) hold for the sheaves $R^i g_*(K_X \otimes P_\eta)$ as well, and the argument goes through without any change, proving (a).

(b) A component of $V^r(R^i g_*(K_X \otimes P_\eta))$ is extremal if and only if, in the notation of the statement, $q(Y) - q(Z) = r$. From (a) we have also the basic inequality $\dim Y - \dim Z \geq r$. Since the map alb_Y is finite, also its restriction to the fibers of h must be finite. Hence the fibers of h surject on the fibers of the homomorphism $\text{Alb } Y \rightarrow \text{Alb } Z$. Therefore $\dim Y - \dim Z = q(Y) - \dim q(Z) = r$. Conversely, since $R^i g_*(K_X \otimes P_\eta)$ is a GV sheaf, the codimension of a component of $V^r(R^i g_*(K_X \otimes P_\eta))$ can't be smaller than r . Therefore for every pair (h, α) as in the statement the translated subtorus $h^* \text{Pic}^0 Z + \alpha$ is a component of $V^r(R^i g_*(K_X \otimes P_\eta))$, in fact an extremal one. \square

6. BACKGROUND MATERIAL: COMPARING CHEN-JIANG DECOMPOSITIONS

In view of Lemma 5.2(b), to study extremal components for sheaves $R^i g_*(K_X \otimes P_\eta)$ as above, we are led to consider commutative diagrams as follows

$$(6.1) \quad \begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb } X \\ \downarrow g & \searrow f & \downarrow \pi_{\text{Alb } Y} \\ Y & \xrightarrow{\text{alb}_Y} & \text{Alb } Y \\ \downarrow h & \searrow l & \downarrow \pi \\ Z & \xrightarrow{a} & B \end{array}$$

where X is compact K hler, the vertical maps on the right are morphisms of complex tori with connected fibers, the vertical maps on the left are Stein factorizations, and the lower part of the

⁶in brief: one can define more generally loci $V_m^i(K_X \otimes P_\eta) = \{\alpha \in \text{Pic}^0 X \mid h^i(K_X \otimes P_\eta \otimes P_\alpha) \geq m\}$ and the Theorems of Green-Lazarsfeld and Simpson-Wang prove as well that all components $V_m^i(K_X \otimes P_\eta)$ are translates of subtori by points of finite order. Then one proves, using Koll r's decomposition, that a component of $V^r(R^i g_*(K_X \otimes P_\eta))$ is also a component of $V_m^{r+i}(K_X \otimes P_\eta)$ for some m .

diagram is such that

$$\dim Y - \dim Z = q(Y) - \dim B := m$$

(from Lemma 5.2(a) it follows also that $B = \text{Alb } Z_B$ and a_B is alb_{Z_B} but we will stick to the notation of (8.4)). Given a point of finite order $\eta \in \text{Pic}^0 X$, by Theorem 3.1 we have the Chen-Jiang decomposition

$$(6.2) \quad R^i f_*(K_X \otimes P_\eta) = \bigoplus_k \pi_k^*(\mathcal{F}_k) \otimes P_{\alpha_k}^{-1}$$

where each $\pi_k : \text{Alb } Y \rightarrow B_k$ is a surjective morphism of complex tori with connected fibers. Moreover, given a point of finite order $\alpha \in \text{Pic}^0 Y$, by Theorem 4.3, we have the Chen-Jiang decomposition

$$(6.3) \quad R^m \pi_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\alpha) = \bigoplus_h \sigma_h^*(\mathcal{G}_h) \otimes P_{\gamma_h}^{-1}$$

where $\sigma_h : B \rightarrow C_k$ are surjective morphisms of complex tori with connected fibers. We would like to compare the two decompositions. This is the content of the following useful property, due to Chen and Jiang ([CJ] Lemma 3.7).

Lemma 6.1. *We adopt the notation above. Then*

$$(6.4) \quad R^m \pi_*(R^i f_*(K_X \otimes P_\eta) \otimes P_\alpha) = \bigoplus_{\pi_k^* \text{Pic}^0 B_k + \alpha_k \subset \pi^* \text{Pic}^0 B + \alpha} R^m \pi_*(\pi_k^*(\mathcal{F}_k) \otimes P_{\alpha_k}^{-1} \otimes P_\alpha)$$

and (6.4) coincides with the Chen-Jiang decomposition (6.3) for $j = m$. In particular, for each summand $\pi_k^* \mathcal{F}_k \otimes P_{\alpha_k}^{-1}$ in (6.2), we have that $R^m \pi_*(\pi_k^* \mathcal{F}_k \otimes P_{\alpha_k}^{-1} \otimes P_\alpha) \neq 0$ if and only if $\pi_k^* \text{Pic}^0 B_k + \alpha_k \subset \pi^* \text{Pic}^0 B + \alpha$. If this is the case, up to twist with a line bundle in $\pi^* \text{Pic}^0 B$, $\pi_k^* \mathcal{F}_k = \pi^* \sigma_h^* \mathcal{G}_h$ for a certain Chen-Jiang summand $\sigma_h^* \mathcal{G}_h$ of (6.3).

Proof. Let us denote $\mathcal{H} = R^i f_*(K_X \otimes P_\eta)$. We know that each factor of the Chen-Jiang decomposition (6.3) corresponds to a component of $V^u(R^m \pi_*(\mathcal{H} \otimes P_\alpha))$ of codimension u in $\text{Pic}^0 B$ (including possibly $u = 0$). Since $\dim Y - \dim Z = q(Y) - \dim B = m$, by Kollár decomposition such components induce components of $V^{m+u}(\mathcal{H} \otimes P_\alpha)$ of codimension $m+u$ in $\text{Pic}^0 Y$, i.e. extremal components for \mathcal{H} . By Remark 3.2(b), this means that $\pi^*(\cdot) \otimes P_\alpha^{-1}$ of the Chen-Jiang decomposition (6.3) is part of the Chen-Jiang decomposition (6.2). With this in mind, let us apply $R^m \pi_*(\cdot \otimes P_\alpha)$ to (6.2). We get another Chen-Jiang decomposition of $R^m \pi_*(\mathcal{G} \otimes P_\alpha)$, essentially coinciding with (6.3) (Remark 3.5). Applying $\pi^*(\cdot) \otimes P_\alpha^{-1}$ we obtain a Chen-Jiang decomposition of a part of (6.2). By uniqueness of Chen-Jiang decompositions (Remark 3.5 again), this essentially coincides with the right hand side of (6.4). Namely it cannot happen that $R^m \pi_*(\pi_k^* \mathcal{F}_k \otimes P_{\alpha_k}^{-1} \otimes P_\alpha) \neq 0$ if $\pi_k^* \text{Pic}^0 B_k + \alpha_k$ is not contained in $\pi^* \text{Pic}^0 B + \alpha$. \square

7. PROOF OF THEOREM C

As it is technically easier, we prove directly Theorem C, before proving Theorem A.

Proof. (a) Let $g : X \rightarrow Y$ be an irregular fibration with

$$m := \dim X - \dim Y > p.$$

For all $\eta \in \text{Pic}^0(g)$ the coherent sheaf $R^m g_*(K_X \otimes P_\eta)$ is a non-zero and GV. Therefore $V^0(R^m g_*(K_X \otimes P_\eta))$ is non-empty by non-vanishing (Theorem 2.2). By Kollár decomposition (Theorem 4.1) and projection formula we have that if $V^0(Y, R^m g_*(K_X \otimes P_\eta))$ was positive-dimensional then also $V^m(X, K_X \otimes P_\eta)$ would be positive-dimensional, against the definition of p . Therefore we are left with the case when $V^0(Y, R^m g_*(K_X \otimes P_\eta))$ is zero-dimensional and the statement of (a) will be proved as soon as we show that in this case: (i) Y is bimeromorphic to a complex torus, and (ii) $\text{Pic}^0 Y = g^* \text{Pic}^0 Y$. We prove (ii) first. By Corollary 3.3(a), a 0-dimensional component of $V^0(R^m g_*(K_X \otimes P_\eta))$ is also a component of $V^{q(Y)}(R^{d(g)} g_*(K_X \otimes P_\eta))$. Again by Kollár decomposition this induces a component of $V^{m+q(Y)}(K_X)$, which is impossible unless $m + q(Y) \leq \dim X$. But $m = \dim X - \dim Y$ and $\dim Y \leq q(Y)$. Therefore $\dim Y = q(Y)$ and, in conclusion, $m + q(Y) = \dim X$. But, since $V^{\dim X}(K_X) = \{\hat{0}\}$, the above is possible only when $\eta \in g^* \text{Pic}^0 Y$. This proves (ii).

It remains to prove (i). Since the loci $V^i(K_X)$ are bimeromorphic invariants of compact Kähler manifolds, after desingularizing Y and replacing X with a suitable bimeromorphic compact Kähler manifold, we can assume that Y is a compact complex manifold. We can assume also that Y is Kähler (e.g. [Ca2] 1.9). Therefore $R^m g_* K_X = K_Y$ ([K1] Prop. 7.6. See also [T] Th. 6.10(iii) for the analytic setting) and, by the above, $V^0(K_Y)$ is 0-dimensional. But a well known result of Ein-Lazarsfeld ([CH1]), tells that this is the case if and only if Y is bimeromorphic to a complex torus. For the reader's convenience, we outline the proof: if $V^0(K_Y)$ is 0-dimensional then, by Remark 3.2, the complex tori B_k corresponding to the Chen-Jiang summands of $\text{alb}_{Y*}(K_Y)$ are 0-dimensional. Therefore $\text{alb}_{Y*}(K_Y)$ would be the direct sum of topologically trivial line bundles on $\text{Alb } Y$. But, since $V^{q(Y)}(Y, K_Y) = \{\hat{0}\}$ (recall that $q(Y) = \dim Y$), it must be $\text{alb}_{Y*} K_Y = \mathcal{O}_{\text{Alb } Y}$. Hence alb_Y is bimeromorphic.

(b) Thanks to Lemma 5.1 and (a) all positive-dimensional components of $V^p(K_X)$ are translates of subtori of the form $g^* V_0(R^p g_*(K_X \otimes P_\eta)) = g^* \text{Pic}^0 Y$ for irregular fibrations $g : X \rightarrow Y$ with $\dim X - \dim Y = p$. Therefore the restriction of P_η to a general fiber of g must be trivial, that is η belongs to $\text{Pic}^0(g)$. It remains to prove that all components of $\text{Pic}^0(g)$ (respectively: all components of $\text{Pic}^0(g)$ but $g^* \text{Pic}^0 Y$ if Y is bimeromorphic to a complex torus) are as above. One proceeds as in (a). To begin with, we claim that for every irregular fibration g as above, every component of $\text{Pic}^0(g) - g^* \text{Pic}^0 Y$ is contained in $V^p(K_X)$ (hence, as it is easy to see, it is a *component* of $V^p(K_X)$). This follows from Lemma 5.1 and (a) as well, because in any case $V^0(R^p g_*(K_X \otimes P_\eta))$ is non-empty (Lemma 2.2). Thus, as in the proof of (a), if P_η does not belong to $g^* \text{Pic}^0 Y$ then $V^0(R^p g_*(K_X \otimes P_\eta))$ is positive-dimensional. But a component of codimension j of $V^0(R^p g_*(K_X \otimes P_\eta))$ is also a component of $V^j(R^p g_*(K_X \otimes P_\eta))$ (Corollary 3.3). Therefore, by Kollár's decomposition, it induces a positive-dimensional component of $V^{p+j}(K_X)$, which contradicts the definition of p . This proves what claimed. By the same reason, either $V^0(R^p g_*(K_X))$ is the full $\text{Pic}^0 Y$ or it is zero-dimensional. In the former case also $g^* \text{Pic}^0 Y$ is a component of $V^p(K_X)$. In the latter case, as in (a), Y must be bimeromorphic to a complex torus. \square

8. THE STANDARD PART

In this section we will prove Theorem A and its Corollaries B and D.

Proof. (of Theorem A) Standard pairs (W, i) arise from irregular fibrations $g : X \rightarrow Y$ with

$$\dim X - \dim Y = i$$

They sit in the usual diagram

$$(8.1) \quad \begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb } X \\ \downarrow g & \searrow f & \downarrow \pi_{\text{Alb } Y} \\ Y & \xrightarrow{\text{alb}_Y} & \text{Alb } Y \end{array}$$

We claim that the components W are of the form

$$W = g^*V^0(R^i g_*(K_X \otimes P_\eta)) + \eta = g^*\text{Pic}^0 Y + \eta$$

for *all* line bundles P_η such that $V^0(R^i g_*(K_X \otimes P_\eta)) = \text{Pic}^0 Y$. As we know, this means that $\chi(R^i g_*(K_X \otimes P_\eta)) > 0$ or, what is the same,

$$(8.2) \quad \chi(R^i f_*(K_X \otimes P_\eta)) > 0.$$

Indeed the components W are as above by Lemma 5.1. Conversely, by the same Lemma, $g^*\text{Pic}^0 Y + \eta$ is contained in $V^i(K_X)$ for such line bundles P_η and it is in fact a component, otherwise it would be strictly contained in a component, say U , of $V^i(K_X)$. But by Lemma 5.1 the component U would arise from another irregular fibration $g' : X \rightarrow Y'$ factoring g , and this is impossible since $\dim X - \dim Y'$ would be strictly smaller than $i = \dim X - \dim Y$, against Lemma 5.1.

Furthermore, since $i = \dim X - \dim Y$, $\eta \in \text{Pic}^0(g)$ if and only if the GV sheaf $R^i g_*(K_X \otimes P_\eta)$ is non-zero, equivalently $R^i f_*(K_X \otimes P_\eta)$ is non-zero. However it can happen that

$$(8.3) \quad \chi(R^i f_*(K_X \otimes P_\eta)) = 0$$

Therefore, recalling the remark preceding the statement of Theorem A, our task consists precisely in describing the subset N_g whose elements are classes $[\eta] \in \text{Pic}^0(g)/g^*\text{Pic}^0 Y$ such that (8.3) holds. Theorem A is equivalent to the following

Claim 8.1. *N_g is either empty or a subgroup of $\text{Pic}^0(g)/g^*\text{Pic}^0 Y$. The latter case happens if and only if $\chi(K_{\tilde{Y}}) = 0$, where \tilde{Y} is a desingularization of Y . If furthermore the Albanese morphism of Y is surjective (i.e. $\dim Y = q(Y)$) then $N_g = \{0\}$.*

As in the proof of Theorem C, we remark that, since the loci $V^i(K_X)$ are bimeromorphic invariants, we can replace the fibration $g : X \rightarrow Y$ with a fibration $\tilde{g} : X' \rightarrow \tilde{Y}$, where $\tilde{Y} \rightarrow Y$ is a Kähler desingularization and X' is Kähler and bimeromorphic to X . Since $R^i \tilde{g}_*(K_{X'}) = K_{\tilde{Y}}$ ([K1] Prop. 7.6, [T] Th. 6.10(iii)) the second sentence of Claim 8.1 follows from the first one.

The first and third sentences of Claim 8.1 are proved following the ideas of Chen and Jiang [CJ]. To this purpose, we consider the finite set of all proper (i.e. strictly contained) subtori of $\text{Pic}^0 Y$ of the form

$$T = \pi_B^* \text{Pic}^0 B$$

where $\pi_B : \text{alb } Y \rightarrow B$ is a surjective homomorphism with connected fibres, such that some translates of T are extremal components of $R^i f_*(K_X \otimes P_\eta)$ for some $\eta \in \text{Pic}^0(g)$. Equivalently, they

are all proper subtori of $\text{Pic}^0 Y$ which are dual to quotients $\text{Alb } Y \rightarrow B_k$ appearing in the Chen-Jiang decomposition of $R^i f_*(K_X \otimes P_\eta)$ for some $\eta \in \text{Pic}^0(g)$. For easy reference, we will call them *extremal subtori* of $\text{Pic}^0 Y$ ⁷. We consider also the usual diagram with the Stein factorizations

$$(8.4) \quad \begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb } X \\ \downarrow g & \searrow f & \downarrow \pi_{\text{Alb } Y} \\ Y & \xrightarrow{\text{alb}_Y} & \text{Alb } Y \\ \downarrow h_B & \searrow l_B & \downarrow \pi_B \\ Z_B & \xrightarrow{a_B} & B \end{array}$$

Given a pair (η, T) , where $\eta \in \text{Pic}^0(g)$ and T is an extremal subtorus of $\text{Pic}^0 Y$, we have $\dim Y - \dim Z_B = q(Y) - \dim B := m_B$ (Corollary 3.3 and Lemma 5.2(b)). We consider the finite set, depending on $[\eta] \in \text{Pic}^0 X / g^* \text{Pic}^0 Y = \Gamma_g$ and T ,

$$\mathcal{N}_T([\eta]) = \{[\alpha] \in \text{Pic}^0(Y)/T \mid R^{m_B} h_{B*}(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha) \neq 0\}$$

We have that:

(a) *the set of $[\eta] \in \Gamma_g$ such that $\mathcal{N}_T([\eta])$ is non-empty is a subgroup $\Sigma_{T,g} \leq \Gamma_g$.*
This is because the condition

$$R^{m_B} h_{B*}(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha) \neq 0$$

is equivalent, by projection formula, to

$$\eta + g^*(\alpha) \in \text{Pic}^0(h_B \circ g)$$

i.e. $\eta \in \text{Pic}^0(g) \cap (\text{Pic}^0(h_B \circ g) + g^* \text{Pic}^0 Y)$.

(b) *for $\eta \in \Sigma_{T,g}$ the set $\mathcal{N}_T([\eta])$ is in bijection with the finite group $\text{Pic}^0(h_B)/T$.*
To prove this, note that the group $\text{Pic}^0(h_B \circ g)$ sits in an extension as follows

$$0 \rightarrow g^* \text{Pic}^0(h) \rightarrow \text{Pic}^0(h_B \circ g) \rightarrow \Gamma'_{h_B, g} \rightarrow 0$$

and

$$(8.5) \quad \text{Pic}^0(h_B \circ g) \cap g^* \text{Pic}^0(Y) = g^* \text{Pic}^0(h_B).$$

(b) follows immediately from (8.5) because, given two elements $[\alpha]$ and $[\beta]$ in $\mathcal{N}_T([\eta])$, $g^*(\alpha - \beta) \in \text{Pic}^0(h_B \circ g) \cap g^* \text{Pic}^0 Y$.

Claim 8.2. *If N_g is not empty, then it is the intersection of the subgroups $\Sigma_{T,g}$ for all maximal (with respect to inclusion) extremal subtori T of $\text{Pic}^0 Y$.*

This proves Claim 8.1, hence the Theorem. Indeed if alb_Y is surjective (equivalently, $\dim Y - q(Y)$) then the trivial subtorus, denoted $\widehat{0}$, is a component of $V^{q(Y)}(Y, K_Y) = V^{q(Y)}(\text{Alb } Y, \text{alb}_{Y,*}(K_Y))$, hence an extremal component of $K_Y = R^i f_*(K_X)$. Therefore $\widehat{0}$ is an extremal subtorus of $\text{Pic}^0 Y$, and it is clear that $\Sigma_{\widehat{0}, g}$ is zero.

⁷we recall (see the footnote to Remark 3.5) that, unlike the homomorphisms π_B , the subtori T are uniquely determined by the Chen-Jiang decomposition

Proof. (of Claim 8.2) We recall that for $\eta \in \text{Pic}^0(g)$ the (generic) rank of $R^i g_*(K_X \otimes P_\eta)$ is equal to one. Hence the rank of $R^i f_*(K_X \otimes P_\eta)$ is equal to $\deg \text{alb}_Y$. In turn $R^i f_*(K_X \otimes P_\eta)$ is the direct sum its Chen-Jiang summands. These are of two types: the M-regular one and the other summands, arising, according to Lemma 5.2(b) from proper extremal subtori $T = \pi_B^* \text{Pic}^0 B$ (see also Remark 3.2, Corollary 3.3 and Lemma 5.2). Therefore, for each $[\eta] \in \Gamma_g = \text{Pic}^0(g)/g^* \text{Pic}^0 Y$

$$(8.6) \quad \deg \text{alb}_Y = H([\eta]) + K([\eta])$$

where H and K denote respectively the rank of the M-regular factor and the sum of the ranks of the other factors. We assert that:

(c) *The integer $K([\eta])$ is maximal if and only if $[\eta]$ belongs to the intersection of the subgroups $\Sigma_{T,g}$ for all proper extremal subtori T .*

To prove (c) we note that, by Lemma 6.1, every non-M-regular Chen-Jiang factor of $R^i f_*(K_X \otimes P_\eta)$ is the pullback of a Chen-Jiang summand of $R^{m_B} \pi_{B*}(R^i f_*(K_X \otimes P_\eta) \otimes P_\alpha)$ for some proper extremal subtorus $T = \pi_B^* \text{Pic}^0 B$ and $[\alpha] \in N_T([\eta])$. If this is the case, we say that such Chen-Jiang summand *belongs to T* (note that, according to this terminology, a Chen-Jiang factor can belong to more than one extremal subtorus). Again by Lemma 6.1 the part of the Chen-Jiang decomposition of $R^i f_*(K_X \otimes P_\eta)$ belonging to T is the Chen-Jiang decomposition of

$$(8.7) \quad \bigoplus_{[\alpha] \in N_T([\eta])} \pi_B^*(R^{m_B} \pi_{B*}(R^i f_*(K_X \otimes P_\eta) \otimes P_\alpha)) \otimes P_\alpha^{-1}$$

Since the rank of

$$(8.8) \quad R^{m_B} h_{B*}(R^i g_*(K_X \otimes P_\eta) \otimes P_\alpha) = R^{m_B+i}(h_B \circ g)_*(K_X \otimes g^* \alpha)$$

is equal to one, the rank of each summand in (8.7) is $\deg a_B$ (see the notation of (8.4)), hence it doesn't depend on $[\alpha] \in N_T([\eta])$.

The above, together with (b), shows that the rank of the part of the Chen-Jiang decomposition of $R^i f_*(K_X \otimes P_\eta)$ belonging to T is equal to the integer (independent on $[\eta]$)

$$\deg a_B \cdot |\text{Pic}^0(h_B)/T|$$

if $[\eta] \in \Sigma_{T,g}$, and zero otherwise. This shows that the integer $K([\eta])$ is maximal if and only if $[\eta]$ is in the intersection of the subgroups $\Sigma_{T,g}$ for all extremal subtori T . Indeed, if this is the case, for all extremal subtori T the part of the Chen-Jiang decomposition of $R^i f_*(K_X \otimes P_\eta)$ belonging to T is non-zero, of rank as in (8.8). Conversely, if $[\eta] \notin \Sigma_{T,g}$ for some extremal subtorus T the part of the Chen-Jiang decomposition of $R^i f_*(K_X \otimes P_\eta)$ belonging to T vanishes. A little argument with Lemma 6.1 shows that $K([\eta])$ can't be maximal. This proves (c).

Finally, assume that the group N_g is non-empty. This means that $H([\eta]) = 0$ for some $[\eta] \in \Gamma_g = \text{Pic}^0(g)/g^* \text{Pic}^0 Y$. By (8.6), this is equivalent to the fact $K([\eta])$ is maximal, namely equal to $\deg \text{alb}_Y$. Then Claim 8.2 follows from (c). \square

Corollary B of the Introduction coincides with Claim 8.1.

Corollary D. Since we are assuming $\dim \text{alb}_X(X) = \dim X$ we have that $R^i \text{alb}_{X*}(K_X) = 0$ for $i > 0$. Therefore $V^i(X, K_X) = V^i(\text{Alb } X, \text{alb}_{X*} K_X)$ (hence, as it is well known from the Theorems of Green and Lazarsfeld, K_X is GV). Therefore every component of $V^0(\text{alb}_{X*} K_X)$ is

an extremal component of some $V^j(\text{alb}_{X*}K_X)$, hence it is standard (Lemma 5.2,). Conversely, since $V^0(\text{alb}_{X*}K_X)$ is strictly contained in Pic^0X , the M -regular summand of the Chen-Jiang decomposition of $\text{alb}_{X*}K_X$ is absent. Therefore, by Remark 3.2, every extremal component for K_X is also a component of $V^0(K_X)$. Therefore Theorem A applies.

9. COMPACT KÄHLER MANIFOLDS WITH $q(X) = \dim X$, $p_1(X) \neq 0$ AND $p_3(X) = 2$

This section is devoted to the proof of Theorem E, which is a slightly weaker extension to the compact Kähler setting of a result of Hacon-Pardini in the algebraic case ([HPa1] Th.6.1). In this sort of matters a mayor role is played by multiplication maps

$$(9.1) \quad \bigoplus_{\eta \in W} H^0(X, L \otimes P_\eta) \otimes H^0(X, M \otimes P_\eta^{-1}) \rightarrow H^0(X, L \otimes M)$$

where W is a suitable subvariety of Pic^0X (usually the translate of a subtorus). It is clear that if $h^0(X, L \otimes P_\eta) = k$ and $h^0(M \otimes P_\eta^{-1}) = h$ generically on W then

$$(9.2) \quad H^0(X, L \otimes M) \geq \dim W + k + h - 1$$

Let X be a compact Kähler manifold with $\dim(X) = q(X)$, $p_1(X) \neq 0$ and $p_3(X) = 2$. Note that this implies that $p_1(X) = 1$ $p_2(X) = 2$. Indeed $p_1(X) = 1$ since otherwise, by a repeated use of the map (9.1) for $W = \{\hat{0}\}$, it follows that $p_3(X) > 3$. Moreover $p_2(X) > 1$ since otherwise, by [PPoS] Thm 1 (previously [CH1] in the algebraic case) X would be bimeromorphic to a complex torus. Therefore $p_2(X) = 2$.

Step 9.1. (i) *The Albanese map of X is surjective* (hence, in view of the hypothesis $\dim X = q(X)$, generically finite onto $\text{Alb } X$).

(ii) *X has fibrations onto elliptic curves $g_i : X \rightarrow E_i$ such that $\text{Pic}^0(g_i) = g_i^*\text{Pic}^0E_i \cup (g_i^*\text{Pic}^0E_i + \eta_i)$ for $i = 1, \dots, k$ (hence the points η_i are of order two) and*

$$V^0(K_X) = \{\hat{0}\} \cup \bigcup_{i=1, \dots, k} (g_i^*\text{Pic}^0E_i + \eta_i)$$

Moreover $h^0(K_X \otimes P_\alpha) = 1$ generically on $g_i^\text{Pic}^0E_k + \eta_i$ for all $i = 1, \dots, k$.*

Proof. In the first place we claim that the origin $\hat{0}$ is an isolated point of $V^0(X)$. As it is well known by [EL], this implies that the Albanese map of X is surjective (proof: since $V^0(X, K_X) = V^0(\text{Alb } X, \text{alb}_{X*}K_X)$, the Albanese map of X is surjective by Corollary 3.3(a)). To prove what claimed, we observe that, for a positive-dimensional subtorus $W \subset V^0(K_X)$, (9.1) and (9.2) for $L = K_X \otimes P_\alpha$, with $\alpha \in W$ and $M = K_X$ would imply that $h^0(K_X^2 \otimes P_\alpha) \geq 2$ for all $\alpha \in W$. Then (9.1) and (9.2) for $L = K_X^2$ and $M = K_X$ and W as above would imply $p_3(X) > 2$.

Now $V^0(K_X)$ is invariant with respect to the natural involution of Pic^0X (by Cor. D, but this was already proved in [CJ] Thm 3.5 and [PPoS], Cor. 16.2). Therefore, given a positive dimensional component W of $V^0(K_X)$, we can consider the map (9.1) with $L = M = K_X$. Since $p_2(X) = 2$ we

get from (9.2) that $\dim W = 1$ and that $h^0(K_X \otimes P_\alpha) = 1$ generically on W . Therefore each of these components is a non-trivial translate of an elliptic curve $\widehat{E_i}$. Dualizing we get maps $g_i : X \rightarrow E_i$:

$$\begin{array}{ccc} X & \xrightarrow{alb_X} & \text{Alb } X \\ & \searrow g_i & \downarrow \pi_i \\ & & E_i \end{array}$$

Next, we prove that the map g_i has connected fibers. In fact $\dim W$ is equal to the genus of the Stein factorization of g_i , say C_i . Therefore $g(C_i) = 1$. Hence g_i has already connected fibers since otherwise it would factorize through an étale map. This would imply that alb_X factorize through an étale map, which is impossible.

Finally, the fact that $\text{Pic}^0(g_i)$ has only two components – or equivalently (by Beauville's theorem mentioned in the Introduction and generalized by Corollary C) there is exactly one component of $V^{\dim X-1}(K_X)$ for each g_i – is as follows. Let $[\eta_i^1], [\eta_i^2] \in \text{Pic}^0(g_i)/g_i^*\text{Pic}^0 E_i$. We consider the following realization of the map (9.1)

$$\bigoplus_{\eta \in \text{Pic}^0 E_i} H^0(X, K_X \otimes P_{\eta_i^1} \otimes P_\alpha \otimes P_\eta) \otimes H^0(X, K_X \otimes P_{\eta_i^2} \otimes P_\eta^{-1}) \rightarrow H^0(X, K_X^2 \otimes P_{\eta_i^1} \otimes P_{\eta_i^2} \otimes P_\alpha),$$

for any $\alpha \in g_i^*\text{Pic}^0 E_i$. By (9.2) this would imply that $H^0(K_X^2 \otimes P_\beta) \geq 2$ for all $\beta \in g_i^*\text{Pic}^0 E_i + \eta_i^1 + \eta_i^2$. Moreover, again by Beauville's theorem, also $g_i^*\text{Pic}^0 E_i - \eta_i^1 - \eta_i^2$ would be a component of $V^0(K_X)$ (unless $[\eta_i^1] = -[\eta_i^2]$). But then the map (9.1) for $L = K_X^2$, $M = K_X$ and $W = g_i^*\text{Pic}^0 E_i + \eta_i^1 + \eta_i^2$ would imply, via the inequality (9.2), that $p_3(X) \geq 3$. \square

Step 9.2. *Keeping the previous notation, $k = 1$ and $alb_{X*}(K_X) = \mathcal{O}_{\text{Alb } X} \oplus (\mathcal{O}_E(p) \otimes P_\eta)$, where p is a point of E .*

Proof. Each component of $V^0(K_X)$ corresponds to a factor of Chen-Jiang decomposition of $alb_{X*}K_X$:

$$alb_{X*}K_X = \mathcal{O}_{\text{Alb } X} \oplus \bigoplus_{i=1, \dots, k} (\pi_i^* \mathcal{F}_i) \otimes P_{\eta_i}$$

Since the maps $g_i : X \rightarrow E_i$ have already connected fibers, we know from Lemma 6.1 that the \mathcal{F}_j 's (M-regular sheaves on the elliptic curves E_i) are the pullback of the M-regular summands of the Chen-Jiang decomposition of $R^{\dim X-1}g_{i*}(K_X \otimes P_{\eta_i})$. However, since such M-regular sheaves have generic rank equal to one, and they are torsion-free, they must be line bundles of positive degree, one for each elliptic curve E_i . Since $h^0(X, g_i^*(L_i \otimes P_\alpha) \otimes P_{\eta_i}) = 1$ for general $\alpha \in \text{Pic}^0 E_i$, it follows that $\mathcal{F}_i = \mathcal{O}_{E_i}(p_i)$, for a point p_i on E_i . Moreover, since $H^0(K_X^2) = 2$ the map

$$\bigcup_{\alpha \in \text{Pic}^0 E_i} H^0(X, g_i^*(\mathcal{O}_{E_i}(p_i) \otimes P_\alpha) \otimes P_{\eta_i}) \otimes H^0(X, g_i^*(\mathcal{O}_{E_i}(p_i) \otimes P_\alpha^{-1}) \otimes P_{\eta_i}) \rightarrow H^0(X, K_X^2)$$

is surjective. It follows that $H^0(X, K_X^2) = H^0(X, g_i^*\mathcal{O}_{E_i}(2p_i))$ and the bicanonical map (in fact morphism) of X factors through E_i . Therefore there is only one such elliptic curve E_i . \square

REFERENCES

- [A] D. Arapura, Higgs line bundles, Green-Lazarsfeld sets, and maps of Kahler manifolds to curves, *Bull. of the Amer. Math. Soc.* **26** (1992) 310–314
- [B] A. Beauville, Annulation du H^1 pour les fibrés en droites plats, *Complex algebraic varieties* LN 1507, Springer-Verlag 1992, 1–15.
- [BeBlPa] O. Ben-Bassat, J. Block, T. Pantev, Non-commutative tori and Fourier-Mukai duality, *Compositio Math.* **143** (2007) 423–475.
- [Ca1] F. Catanese, Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations, *Invent. Math.* **104** (1991), 389–407.
- [Ca2] F. Catanese, Compact complex manifolds bimeromorphic to tori, *Abelian Varieties*, De Gruyter (1995) 55–62
- [CH1] J.A. Chen and Ch. Hacon, Characterization of abelian varieties, *Invent. Math.* **143** (2001) 435–447.
- [CH2] J.A. Chen and Ch. Hacon, On algebraic fiber spaces over varieties of maximal Albanese dimension, *Duke Math. J.* **111** (2002), 159–175.
- [CH3] J.A. Chen and Ch. Hacon, Varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **III** (2004), no. 2, 399–425.
- [CJ] J. A. Chen, Z. Jiang, Positivity in varieties of maximal Albanese dimension, to appear on *J. Reine Angew. Math.*. Published online: DOI: 10.1515/crelle-2015-0027 (2015)
- [EL] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, *J. Amer. Math. Soc.* **10** (1997), no.1, 243–258.
- [GL1] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, *Invent. Math.* **90** (1987), 389–407.
- [GL2] M. Green and R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, *J. Amer. Math. Soc.* **1** (1991) 87–103.
- [H1] Ch. Hacon, A derived category approach to generic vanishing, *J. Reine Angew. Math.* **575** (2004), 173–187.
- [H2] Ch. Hacon, Varieties with $P_3 = 3$ and $q(X) = \dim(X)$, *Math. Nachr.* **278** (2005) 409–420
- [HPa1] C. Hacon and R. Pardini, On the birational geometry of varieties of maximal Albanese dimension, *J. Reine Angew. Math.* **546** (2002) 177–199.
- [HPa2] C. Hacon and R. Pardini, Birational characterization of products of curves of genus 2, *Math. Res. Lett.* **12** (2005) 129–140
- [Hu] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Clarendon Press - Oxford (2006)
- [J] Z. Jiang, Varieties with $q(X) = \dim(X)$ and $P_2(X) = 2$, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **XI** (2012) 243–258
- [K1] J. Kollár, Higher direct images of dualizing sheaves I, *Ann. of Math.* **123** (1986), 11–42.
- [K2] J. Kollár, Higher direct images of dualizing sheaves II, *Ann. of Math.* **124** (1986), 171–202.
- [LPo] L. Lombardi, M. Popa, Derived equivalence and non-vanishing loci II, *London Math. Soc. Lecture Note Series*, 417, *Recent Advances in Algebraic Geometry*, Cambridge University Press (2015) 291–306
- [M] S. Mukai Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153–175.
- [P] G. Pareschi, Basic results on irregular varieties via Fourier-Mukai methods, *Current developments in algebraic geometry*, Math. Sci. Res. Inst. Publ., vol. 59, Cambridge Univ. Press, Cambridge, 2012, pp. 379–403.
- [PPo3] G. Pareschi and M. Popa, Regularity on Abelian Varieties III: Relationship with Generic Vanishing and Applications, *Grassmannians, Moduli Spaces and Vector Bundles*, AMS 2011, 141–167
- [PPoS] G. Pareschi, M. Popa and Ch. Schnell, Hodge modules on complex tori and generic vanishing for compact Kähler manifolds preprint arXiv:1505.00635v1 [math.AG]
- [Po] M. Popa, Derived equivalence and non-vanishing loci, *Clay Mathematical Proceedings* 18, AMS 2013, 567–575
- [Sa1] M. Saito, Decomposition theorem for proper Kahler morphisms, *Tohoku Math. J. (2)* **42** (1990), 127–147.
- [Sa2] M. Saito, On Kollár’s conjecture, *Several complex variables and complex geometry*, Proc. Sympos. Pure Math., vol. 52, AMS 1991, 509–517.
- [S] Ch. Schnell, Torsion points on cohomology support loci: from D-modules to Simpson’s theorem, *Recent advances in algebraic geometry*, London Math. Soc. Lecture Note Ser. 417 (2015), 405–421
- [Si] C. T. Simpson, Subspaces of moduli spaces of rank one local systems, *Ann. Sci. École Norm. Sup.* **4 26** (1993) 361–401

- [T] K. Takegoshi, Higher direct images of canonical sheaves tensorized by semi-positive vector bundles by proper Kahler morphisms, Math. Ann. **303** (1995) 389-416
- [W] B. Wang, Torsion points on the cohomology jump loci of compact Kähler manifolds, Math. Res. Lett. **23** (2016) 545–563.

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